

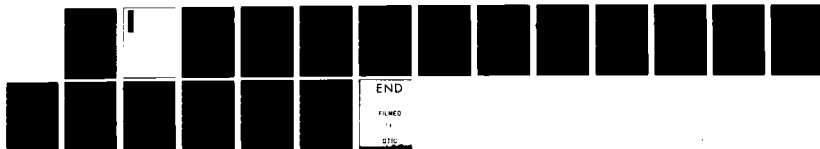
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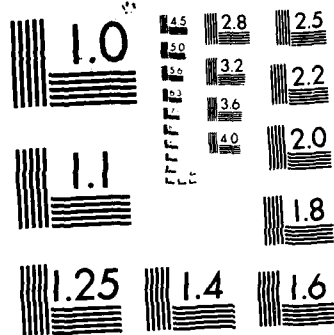
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ESTIMATION IN NONCENTRAL DISTRIBUTIONS

By
Alan E. Gelfand

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ESTIMATION IN NONCENTRAL DISTRIBUTIONS

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1. INTRODUCTION

This paper investigates alternatives to minimum variance unbiased (MVU) estimators in noncentral (NC) χ^2 and F distributions. Motivation is provided by noting that if $Z \sim \chi^2(p; \lambda)$, then i) $(Z-p)/2$, the MVU for $g(\lambda) = \lambda$ is inadmissible under squared error loss (SEL), see, e.g., Perlman and Rasmussen (1975). ii) Z^{-1} , the MVU for $g(\lambda) = E(Z^{-1})$ is inadmissible under any "bowl shaped" loss since $S(Z) = \min(Z^{-1}, (p-2)^{-1})$ dominates. Similar conclusions hold when $W \sim F(p, r; \lambda)$. Case (i) is of obvious interest. Case (ii) arises in estimating the improvement under SEL of the James-Stein estimator of the multivariate normal mean, see, e.g., Efron and Morris (1976).

Two directions will be pursued. In the first a simple approach for uniformly improving upon MVU estimators is described and illustrated. In the second Bayesian procedures are characterized and illustrated. This effort extends earlier work of Perlman and Rasmussen (1975), Neff and Strawderman (1976), and DeWaal (1974).

2. DISTRIBUTION THEORY AND NOTATION

In the NC χ^2 case, the joint density $f(Z, L, \lambda)$ arises from $f(Z|L, \lambda) = f(Z|L) = \chi^2(p+2L)$, $f(L|\lambda) = \text{Po}(\lambda)$ (Poisson) and a prior $\tau(\lambda)$ on λ . If $\lambda \sim \gamma/2 \chi^2(p)$ (arising from $\lambda = \mu^T \mu/2$ with $\mu \sim N_p(0, \gamma I)$ as discussed in, e.g., Perlman and Rasmussen (1975)), $f(\lambda|L, Z) = f(\lambda|L) = \rho/2 \chi^2(p+2L)$, where $\rho = \gamma(\gamma+1)^{-1}$, $f(\gamma|Z) = \rho/2 \chi^2(p, \rho Z/2)$, $f(L) = \text{NB}(p/2, \rho)$ (negative binomial with mean $p\rho(1-\rho)^{-1}/2$), $f(Z) = (\gamma+1)\chi^2(p)$ and $f(L|Z) = \text{Po}(\rho Z/2)$.

If $W = Z/U$ where $Z \sim \chi^2(p; \lambda)$ independent of $U \sim \chi^2(r)$, W has a nonnormalized NC F distribution. The joint density $f(W, L, \lambda)$ arises from $f(W|L, \lambda) = f(W|L) = r^{-1}(p+2L)F(p+2L, r)$, $f(L|\lambda)$ and $\tau(\lambda)$ as above.

The fact that $f(Z|L, \lambda) = f(Z|L)$, $f(W|L, \lambda) = f(W|L)$ is useful in improving upon MVU's. The fact that regardless of τ , $f(\lambda|L, Z) = f(\lambda|L)$, $f(\lambda|L, W) = f(\lambda|L)$ is useful for finding Bayes estimators. In this vein, if $g(\lambda)$ is to be estimated, let $b(L) = E(g(\lambda)|L)$, $a(L)$ be such that $g(\lambda) = E(a(L)|\lambda)$. Then $a(L)$ helps to find improved estimators, $b(L)$ helps to find Bayes estimators.

In Sections 3 and 4 the methodology is examined through a collection of examples. A summary table is given at the end to unify the findings.

3. FINDING IMPROVED ESTIMATORS

We describe a simple approach which has been successful in creating classes of estimators that uniformly improve upon the MVU of $g(\lambda)$ under SEL. We then illustrate with several examples. For the NC χ^2 case, let $T(Z)$ be the MVU of $g(\lambda)$. When will $T(Z) + \phi(Z)$

uniformly improve upon $T(Z)$? The conditional estimation problem, estimating $a(L)$, provides sufficient conditions while enabling us to work with the simpler central χ^2 distribution. We note that T is unbiased for $a(L)$. By direct calculation we can show that the difference in SEL between T and $T+\phi$ is:

$$-E(I_\phi(L)|\lambda) - 2 \text{ cov}(a(L), \phi(Z)|\lambda) \quad (1)$$

where

$$I_\phi(L) = E(\phi^2|L) + 2E((T-a)\phi|L) \quad (2)$$

and the covariance is with respect to the joint distribution of L and Z given λ .

Hence if

$$I_\phi(L) \leq 0 \quad \forall L \quad (3a)$$

with strict inequality for some L and

$$\text{cov}(a, \phi) \leq 0, \quad (3b)$$

then $T+\phi$ will dominate T . If $a(L)$ is monotone, restriction of ϕ such that $E(\phi|L)$ is nonincreasing in $a(L)$ will satisfy the covariance condition. If the range of $g(\lambda)$ is a subinterval of R^1 , $T+\phi$ should be similarly restricted. In particular, if $g(\lambda) \geq 0$, the positive part estimator, $[T+\phi]^+$, dominates $T+\phi$.

For the NC F all the above remarks apply with Z replaced by W .

Examples: NC χ^2

Two convenient choices of ϕ are $\phi_{\alpha, \beta}^{(1)} = \alpha Z^\beta$, $\phi_{\alpha, \beta}^{(2)} = \alpha e^{\beta Z}$ in which case the fact that, $n < 1/2$, $m > -p/2$,

$$E(Z^m e^{nZ}|L) = \frac{\Gamma(p/2 + L + m)}{\Gamma(p/2 + L)} \frac{2^m}{(1-2n)^{p/2 + L + m}}$$

will be helpful. We assume $p \geq 5$.

1) If $g(\lambda) = \lambda$, $T(Z) = (Z-p)/2$ and $b(L) = L$. For $\phi^{(1)}$ (3b) will be satisfied if $\alpha < 0$, $\beta > 0$ or $\alpha > 0$, $-p/2 < \beta < 0$. For $\phi^{(2)}$ (3b) will be satisfied if $\alpha < 0$, $0 < \beta < 1/2$ or $\alpha > 0$, $\beta < 0$. Clearly, for any α, β

$$\text{cov}(L, \phi^{(i)}) \rightarrow 0 \text{ as } \lambda \rightarrow 0, i = 1, 2. \quad (4)$$

If $\beta > -p/4$, $I_{\phi(1)}(L) = 2^\beta \alpha (\Gamma(p/2 + L))^{-1} \cdot (2^\beta \alpha \Gamma(p/2 + L + 2\beta) + 2\beta \Gamma(p/2 + L + \beta))$. Hence $-p/4 < \beta < 0$ requires $0 < \alpha < -2^{1-\beta} \beta \Gamma(p/2 + L + \beta) \cdot (\Gamma(p/2 + L + 2\beta))^{-1} \forall L$. Since the right-hand side is smallest at $L = 0$, we obtain the condition $0 < \alpha < -2^{1-\beta} \beta \Gamma(p/2 + \beta) (\Gamma(p/2 + 2\beta))^{-1}$ in agreement with Neff and Strawderman (p. 66) and including Perlman and Rasmussen (p. 464). If α exceeds this condition $I_{\phi(1)}(0) > 0$, i.e., with (4), $T + \phi^{(1)}$ can't dominate T . If $\beta > 0$, we require $0 > \alpha > -2^{1-\beta} \beta \Gamma(p/2 + L + \beta) \cdot (\Gamma(p/2 + L + 2\beta))^{-1} \forall L$ which is impossible as $L \rightarrow \infty$. Rasmussen (1973) shows that when $\beta = 1$ no ϕ of the form $\alpha Z + \delta$ yields $T + \phi$ which dominates T , i.e., no linear estimator can dominate T .

If $\beta < 1/4$, $I_{\phi(2)}(L) = \alpha(1-4\beta)^{-(p/2 + L)} \cdot \alpha + 2\beta(1-2\beta)^{-1}(p+2L)\eta_1(L)$ where $\eta_1(L) = [(1-4\beta)/(1-2\beta)]^{p/2 + L}$. Similar to the $\phi^{(1)}$ case, $\beta < 0$ requires $0 < \alpha < -2p\beta(1-2\beta)^{-1} \eta_1(0)$ to have $I_{\phi(2)} < 0 \forall L$. If α exceeds this condition, with (4), $I_{\phi(2)}(0) > 0$, i.e. $T + \phi^{(2)}$ can't dominate T . If $0 < \beta < 1/4$, no $\alpha < 0$ works if L is sufficiently large.

ii) If $g(\lambda) = E(Z^{-1})$ (up to a constant $g(\lambda)$ is the improvement of the James-Stein estimator of the multivariate normal mean when the variance is known), $T(Z) = Z^{-1}$, $b(L) = (p-2+2L)^{-1}$. For $\phi^{(1)}$ (3b) holds if $\alpha < 0$, $-p/2 < \beta < 0$ or $\alpha > 0$, $\beta > 0$. For $\phi^{(2)}$, (3b) holds if $\alpha < 0$, $\beta < 0$ or $\alpha > 0$, $0 < \beta < 1/2$.

If $\beta > -p/4$, $I_{\phi(1)}(L) = 2^\beta \alpha (\Gamma(p/2 + L))^{-1} \cdot (2^\beta \alpha \Gamma(p/2 + L + 2\beta) - 2\beta(p-2+2L)^{-1} \Gamma(p/2 + L + \beta - 1))$. When $p < 8$ with $\beta > -p/4$, (i.e., $\beta > -2$), $2^{1-\beta} \beta(p-2+2L)^{-1} \Gamma(p/2 + L + \beta - 1) (\Gamma(p/2 + L + 2\beta))^{-1} \rightarrow 0$ as $L \rightarrow \infty$; no $T + \phi^{(1)}$ which dominate T are revealed. If $p \geq 8$, $-p/4 \leq \beta < -2$, $0 > \alpha > 2^{1-\beta} \beta(p-2)^{-1} \Gamma(p/2 + \beta - 1) (\Gamma(p/2 + 2\beta))^{-1}$, $T + \phi^{(1)}$ will dominate T .

If $\beta < 1/4$, $I_{\phi(2)}(L) = \alpha(1-4\beta)^{-(p/2 + L)} \cdot (\alpha - 4\beta(p-2+2L)^{-1} \eta_1(L))$. If $\beta < -(p-4)^{-1}$ and $0 > \alpha > 4\beta(p-2)^{-1} \eta_1(0)$ ($\beta < -(p-4)^{-1}$ is needed to insure that $4\beta(p-2+2L)^{-1} \eta_1(L)$ is largest at $L = 0$), $T + \phi^{(2)}$ will dominate T . If $0 < \beta < 1/4$, no $\alpha > 0$ works if L is sufficiently large.

iii) If $g(\lambda) = c^\lambda \Leftrightarrow g(\lambda) = e^{c\lambda}$, $c > -1$, $T(Z) = (1-d)^{p/2} e^{dZ/2}$ where $d = c(c+1)^{-1}$, $a(L) = (c+1)^L$. If, in fact, $c > 0$ we can show that for both $\phi^{(1)}$ and $\phi^{(2)}$ there exist α, β such that $T + \phi^{(1)}$, $T + \phi^{(2)}$ dominate T . We omit the details.

iv) If $g(\lambda) = \lambda^2$, $T(Z) = Z^2/4 - (p+2)Z/2 + p(p+2)/4$ and $a(L) = L(L-1)$. Since a is monotone on the support of L , (3b) will be satisfied for $\phi^{(1)}$ and $\phi^{(2)}$ over the same ranges as in example (i).

If $\beta > -p/4$, $I_{\phi(1)}(L) = 2^\beta \alpha (\Gamma(p/2 + L))^{-1} \cdot (2^\beta \alpha \Gamma(p/2 + L + 2\beta) + 2\beta(\beta + 2L - 1) \Gamma(p/2 + L + \beta))$. It is apparent that if $-p/4 < \beta \leq 0$, $\alpha > 0$, $I_{\phi(1)}(0) > 0$; if $0 < \beta \leq 1$, $\alpha < 0$, $I_{\phi(1)}(0) > 0$. Here $T + \phi^{(1)}$ can't dominate T . When $\beta > 1$, no α works if L is sufficiently large; this approach doesn't reveal any $T + \phi^{(1)}$ which dominate T .

If $\beta < 1/4$, $I_{\phi(2)}(L) = \alpha(1-4\beta)^{-(p/2 + L)} \cdot (\alpha + 2\beta(1-2\beta)^{-2} \eta_1(L) (4L^2(1-\beta) + 2L(p+2\beta) + \beta p(p+2)))$.

Again we do not reveal any $T + \phi^{(2)}$ which improve upon T . However $\phi = \alpha Z^{-2} e^{\beta Z}$, for example, can be used successfully. We omit the details.

Examples: NC F

We illustrate for $\phi_{\alpha, \beta} = \alpha W^{\beta}$. (The more general $\phi = \alpha W^{\beta_1} (1+W)^{\beta_2}$ can be used to broaden the conclusions. We omit the details.) It is useful to note that, $-p/2 < \beta < r/2$.

$$E(W^{\beta} | L) = \frac{\Gamma(p/2 + L + \beta)}{\Gamma(p/2 + L)} \frac{\Gamma(r/2 - \beta)}{\Gamma(r/2)}$$

which increases in L if $\beta > 0$, decreases in L if $\beta < 0$. We assume $p, r \geq 5$.

v) If $g(\lambda) = \lambda$, $T(W) = ((r-2)W-p)/2$ and $a(L) = L$. Condition (3b) will be satisfied if $\alpha > 0$, $-p/2 < \beta < 0$ or $\alpha < 0$, $0 < \beta < r/2$ and for any α, β clearly $\text{cov}(L, \phi) \rightarrow 0$ as $\lambda \rightarrow 0$.

For $-p/4 < \beta < r/4$, $I_{\phi}(L) = \alpha \eta_2(L) \cdot (\alpha + 2\beta \eta_3(L)(p+r+2L-2)(r-2\beta-2)^{-1})$ where

$$\eta_2(L) = \frac{\Gamma(p/2 + L + 2\beta)}{\Gamma(p/2 + L)} \frac{\Gamma(r/2 - 2\beta)}{\Gamma(r/2)},$$

$$\eta_3(L) = \frac{\Gamma(p/2 + L + \beta)}{\Gamma(p/2 + L + 2\beta)} \frac{\Gamma(r/2 - \beta)}{\Gamma(r/2 - 2\beta)}.$$

Note that $\eta_3(L) = O(L^{-\beta})$ and that $(p+r+2L-2)\eta_3(L)$ increases in L for $-p/4 < \beta < p(p+r-4)^{-1}$. Hence $-p/4 < \beta < 0$, $0 < \alpha < -2\beta \eta_3(0)(p+r-2)(r-2\beta-2)^{-1}$ or $0 < \beta < p(p+r-4)^{-1}$, $0 > \alpha > -2\beta \eta_3(0)(p+r-2)(r-2\beta-2)^{-1}$ yields $T+\phi$ which dominates T . For $\beta < 0$, if α exceeds the condition, $I_{\phi}(0) > 0$. The case $\beta = -1$ is discussed in Perlman and Rasmussen (p. 467). At $\beta = 1$, $\eta_3(L) \cdot (p+r+2L-2)(r-2\beta-2)^{-1} = (p+r+2L-2)(p+2L+2)^{-1} > 1$ so that $-2 < \alpha < 0$, $T+\phi$ will dominate T . In fact, if $\phi = \alpha W + \gamma$

a more general dominating family of linear estimators can be created in agreement with Perlman and Rasmussen (p. 465-6).

vi) For $g(\lambda) = E(W^{-1})$ (up to a constant, $g(\lambda)$ is the improvement of the James-Stein estimator when an independent estimator of the variance having r degrees of freedom is used), $T = W^{-1}$ and $a(L) = r(p-2+2L)^{-1}$. Here $\alpha > 0$, $0 < \beta < r/2$ or $\alpha < 0$, $-p/2 < \beta < 0$ will satisfy the covariance condition.

For $-p/4 < \beta < r/4$, $I_\phi(L) = \alpha \eta_2(L)(\alpha - 4\beta \eta_3(L)\eta_4(L))$ where $\eta_4(L) = (p+r+2L-2)(p+2L-2)^{-1}(p+2L+2\beta-2)^{-1}$ and $\eta_3(L) \cdot \eta_4(L)$ increases in L if $-p/4 < \beta < -(p(p-2+2r) \cdot (p(p-2)+r(p+2)))^{-1}$. For β in this range, if $0 > \alpha > 4\beta \eta_3(0)\eta_4(0)$, $T+\phi$ will dominate T . At $\beta = -1$, $\eta_3(L)\eta_4(L) > (r+2)^{-1}$ which requires $0 > \alpha > -4(r+2)^{-1}$. Hence cW^{-1} will dominate W^{-1} when $(r-2)(r+2)^{-1} < c < 1$.

4. BAYES ESTIMATORS

In developing Bayesian procedures under SEL, we again turn to the conditional problem, i.e., Z/L . The relation

$$\pi(L) = \int \frac{\lambda^L e^{-\lambda}}{L!} \tau(\lambda) d\lambda \quad (5)$$

shows which $\pi(L)$ can arise as priors. In fact, since (5) is an instance of the classical moment problem, if $\pi(L)L!$ is a "moment sequence" (see, e.g., Feller (1966) Sec. VII.3 for conditions) $\pi(L)$ uniquely determines $\tau(\lambda)$. A useful case is, $v > -\frac{p-2}{2}$,

$$\begin{aligned} \lambda &\sim G(p/2 + v; \gamma^{-1}) \text{ (Gamma with mean } (p/2 + v)\gamma), \\ L &\sim NB(p/2 + v; \rho) \text{ } (\rho = \gamma(\gamma+1)^{-1}). \end{aligned} \quad (6)$$

Under (6), $\lambda|L \sim G(p/2 + v+L; \rho^{-1})$.

Recalling that $b(L) = E(g(\lambda)|L)$, for the NC χ^2 we have by direct calculation $E(g(\lambda)|Z) = E(b(L)|Z)$, i.e., we can calculate Bayes rules using the central χ^2 distribution. The same conclusions hold for the NC F with W replacing Z.

Examples: NC χ^2

For a particular π the Bayes rule

$$\delta_{\pi}(Z) = E(b(L)|Z) = \frac{\sum b(L)(Z/2)^L \pi(L)(\Gamma(p/2 + L))^{-1}}{\sum (Z/2)^L \pi(L)(\Gamma(p/2 + L))^{-1}}. \quad (7)$$

Denote the denominator in (7) by $J_{\pi}(Z)$.

i) If $g(\lambda) = \lambda$, then from (5) $b(L) = (L+1) \cdot \pi(L+1)(\pi(L))^{-1}$.

Using this, straightforward manipulation yields

$$\delta_{\pi}(Z) = 2Z \frac{J_{\pi}^{(0)}(Z)}{J_{\pi}(Z)} + p \frac{J_{\pi}^{(1)}(Z)}{J_{\pi}(Z)}. \quad (8)$$

Setting $\delta_{\pi}(Z)$ to be the MVU of λ in (8) leads to a second order homogeneous linear differential equation whose general solution is

$$J_{\pi}(Z) = e^{-Z/2} (c_1 + c_2 \int Z^{-p/2} e^Z dz). \quad (9)$$

By the definition of J_{π} c_2 cannot equal 0. But then after multiplying both sides of (9) by $e^{Z/2}$ and differentiating, we see that $c_2 \neq 0$ can't work either. Thus the MVU can't be the limit of Bayes or extended Bayes clarifying its inadmissibility.

Applying the NB priors in (6) to (8) and denoting the resulting rule by $\delta_{v,\rho}(Z)$, we have

$$\delta_{v,\rho}(Z) = \rho \left(\frac{p}{2} + v + \frac{Z J_{\pi}'}{J_{\pi}} \right). \quad (10)$$

It is straightforward to show that for fixed ρ , $v + \frac{ZJ_{\pi}}{J_{\pi}}$ increases in v whence $\delta_{v_1, \rho} < \delta_{v_2, \rho}$ if $v_1 < v_2$. At $v = 0$, $J_{\pi}(Z) = ce^{\rho Z/2}$, $\delta_{0, \rho} = \rho(\rho Z + p)/2$ which is discussed at length (particularly when $\rho = 1$) in Perlman and Rasmussen. Corresponding to the noninformative prior, $\pi(L) = 1$ from $\tau(\lambda) = 1$ (which arises in (6) at $v = -(p-2)/2$, $\rho = 1$), we have $\delta_{-(\frac{p-2}{2}), 1} = \frac{Z-p}{2} + (J_{\pi}(Z)\Gamma(p/2 - 1))^{-1} + 2$ so that the MVU, $T < \delta_{v, 1} \forall v$. In fact, $\delta_{-(\frac{p-2}{2}), 1} = T + \phi$ with ϕ satisfying (3b), but we are unable to show that this δ dominates T . We note that Neff and Strawderman derive a subclass of (8) of proper Bayes estimators for λ arising from a two-stage prior distribution and show that none of these dominate T .

ii) Generally from (5) if $g(\lambda) = \lambda^r$, a positive integer $b(L) = (L+r)_r \pi(L+r)(\pi(L))^{-1}$. In particular, for λ^2 the Bayes rule becomes

$$\delta_{\pi}(Z) = 4Z^2 \frac{J_{\pi}^{(iv)}(Z)}{J_{\pi}(Z)} + 4Z(p+2) \frac{J_{\pi}^{(iii)}(Z)}{J_{\pi}(Z)} + p(p+2) \frac{J_{\pi}^{(ii)}(Z)}{J_{\pi}(Z)}. \quad (11)$$

Using an argument similar to that below (9), it may be possible to show that the MVU can't be the limit of Bayes or extended Bayes. Applying the NB priors in (6) to (11), we obtain, for example, $v = 0$, $\rho^4 Z^2/4 + \rho^3 Z(p+2)/2 + \rho^2 p(p+2)/4$.

iii) If $g(\lambda) = e^{c\lambda}$, under (6) with $c < \rho^{-1}$, $b(L) = (1-\rho c)^{-(p/2 + v + L)}$. Bayes rules take the form

$$\delta_{\pi}(Z) = (1-\rho c)^{-(p/2 + v)} J_{\pi}((1-\rho c)^{-1}Z) / J_{\pi}(Z). \quad (12)$$

At $v = 0$ we obtain $(1-\rho c)^{-p/2} e^{\rho^2 c Z/2(1-\rho c)}$. The inadmissible MVU estimator would arise if ρ could equal -1 .

iv) If $g(\lambda) = E(Z^{-1}) = E((p-2+2L)^{-1}|\lambda)$, then under (6), $b(L) = E((p-2+2M)^{-1}|L)$ where $M|L \sim NB(p/2 + v+L, \rho(\rho+1)^{-1})$, i.e.,

$$b(L) = \sum_{m=0}^{\infty} \frac{\Gamma(p/2 + v+L+m)}{\Gamma(p/2 + v+L)m!} \cdot (p-2+2m)^{-1} \left(\frac{\rho}{\rho+1}\right)^m \left(\frac{1}{\rho+1}\right)^{p/2 + v+L}.$$

When v is a nonnegative integer, we have the identity

$$\begin{aligned} & \sum_{m=0}^{\infty} \frac{\Gamma(p/2 + v+L+m)}{\Gamma(p/2 + v+L)m!} \cdot (p-2+2m)^{-1} \left(\frac{\rho}{\rho+1}\right)^{p/2 + m-1} \\ &= \sum_{n=0}^{v+L} \binom{v+L}{n} (p-2+2n)^{-1} \rho^{p/2 + n-1} \end{aligned}$$

(derivable by considering the indefinite integral with respect to ρ of $\rho^{p/2 - 2}(\rho+1)^{L+v}$ directly and through its equivalent negative binomial expansion). This yields the Bayes rule in the form

$$\begin{aligned} \frac{(1-\rho)^{p/2}(\rho+1)^{-(v+1)}}{J_{\pi}(Z)\Gamma(p/2 + v)} \cdot \sum_{L=0}^{\infty} \frac{\Gamma(p/2 + L+v)}{\Gamma(p/2 + L)L!} \left(\frac{\rho}{\rho+1} \frac{Z}{2}\right)^L \\ \cdot \sum_{n=0}^{v+L} \binom{v+L}{n} (p-2+2n)^{-1} \rho^n. \end{aligned} \quad (13)$$

At $v = 0$, interchanging order of summation in (13), we obtain the rule as $(\rho+1)^{-1} E((p-2+2L)^{-1}|Z) = (\rho+1)^{-1} \cdot g(\rho^2 Z/2(\rho+1))$ since $L|Z$ has a Poisson distribution.

Examples: NC F

For $g(\lambda)$, $b(L)$, under π , the Bayes rule becomes (in terms of $V = W(1+W)^{-1} \sim NC \text{ Beta}$)

$$\delta_{\pi}(V) = E(a(L)|V) = \frac{\sum b(L) V^L \pi(L) \Gamma(\frac{p+r}{2} + L) (\Gamma(\frac{p}{2} + L))^{-1}}{\sum V^L \pi(L) \Gamma(\frac{p+r}{2} + L) (\Gamma(\frac{p}{2} + L))^{-1}}. \quad (14)$$

Denote the denominator in (14) by $K_{\pi}(V)$.

v) If $g(\lambda) = \lambda$ it is straightforward to show that

$$\delta_{\pi}(V) = (K_{\pi}(V))^{-1} V^{-\frac{(p+r)}{2}} \int V^{\frac{p+r}{2}-1} (VK_{\pi}^{(n)}(V) + \frac{p}{2} K_{\pi}^{(1)}(V)). \quad (15)$$

Under (6), (15) becomes (denoting the rule by $\delta_{v,\rho}$)

$$\delta_{v,\rho}(V) = \rho[p/2 + v + VK_{\pi}'/K_{\pi}]. \quad (16)$$

Note the similarity between (16) and (10). It is straightforward to show that for fixed ρ , $v + VK_{\pi}'/K_{\pi}$ increases in v whence $\delta_{v_1,\rho} < \delta_{v_2,\rho}$ if $v_1 < v_2$. At

$$v = 0, K_{\pi}(V) = (1-\rho)^{p/2} (\Gamma(p/2))^{-1} \Gamma(\frac{p+r}{2}) (1-\rho V)^{-(p+r)/2}$$

so that $\delta_{0,\rho}(V) = \rho/2((p+r)\rho V(1-\rho V)^{-1} + p)$ which is discussed in Perlman and Rasmussen (p. 466), particularly when $\rho = 1$, ($\delta_{0,1} = ((p+r)W+p)/2$). Corresponding to the noninformative prior, we obtain $\delta_{-(\frac{p-2}{2}),1} = (r-2)W/2 - p/2 + (W+1) \cdot (2 + \frac{\Gamma((p+r)/2)}{\Gamma(p/2-1) \cdot K_{\pi}(W(1+W)-1)})$ whence

$T < \delta_{v,1}, \forall v$, where T is the MVU estimator.

vi) If $g(\lambda) = E((p-2+2L)^{-1}|\lambda)$, under (6) calculations analogous to those leading to (13) yield at $v = 0$ the rule $(\rho+1)^{-1} E((p-2+2L)^{-1}|W)$ as in the NC χ^2 case except $L|W$ has a negative binomial distribution.

5. SUMMARY AND DIRECTION FOR FUTURE WORK

We present a summary table of the disparate findings in this article.

Two problems requiring further investigation are the following. First it would be useful to link the two estimation approaches discussed herein. More precisely, writing the Bayes estimators in Section 4 in the form

$T+\phi$, where T is the corresponding MVU, can we use the method of Section 3 (or some other argument) to obtain Bayes rules which uniformly improve upon the MVU? Do any Bayes rules dominate the MVU? Second, under the priors in (6) at, for example, $v = 0$ the marginal distribution of Z is $(1-\rho)^{-1}\chi^2(p)$ suggesting that for the NC χ^2 case convenient "empirical Bayes" estimators can be developed. Do any of these estimators dominate the corresponding MVU? Can the method of Section 3 help in this regard?

Finally, the meticulous and provocative effort of the referee is acknowledged. The paper is much improved as a result.

TABLE I

	$g(\lambda)$	Improving	Bayes Estimator
		Estimator of the Form $T+\phi$	under SEL
NC χ^2	λ	$\phi = \alpha Z^\beta$ $\phi = \alpha e^{\beta Z}$ See Ex. 3(i)	General form: (8) Under priors in (6):(10) See Ex. 4(i)
	λ^2	$\phi = \alpha Z^2 e^{\beta Z}$ See Ex. 3(iv)	General form: (11) See Ex. 4(ii)
	c^λ or $e^{c\lambda}$	$\phi = \alpha Z^\beta$ $\phi = \alpha e^{\beta Z}$ See Ex. 3(iii)	Under priors in (6):(12) See Ex. 4(iii)
	$E(Z^{-1})$	$\phi = \alpha Z^\beta$ $\phi = \alpha e^{\beta Z}$ See Ex. 3(ii)	Under priors in (6):(13) See Ex. 4(iv)
NC F	λ	$\phi = \alpha W^\beta$ See Ex. 3(v)	General form: (15) Under priors in (6):(16) See Ex. 4(v)
	$E(W^{-1})$	$\phi = \alpha W^\beta$ See Ex. 3(vi)	See Ex. 4(vi)

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ESTIMATION IN NONCENTRAL DISTRIBUTIONS

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This paper investigates alternatives to MVU estimators in noncentral χ^2 and F distributions. Two directions are pursued. In the first, a general approach for uniformly improving on MVU estimators is described and illustrated. In the second, Bayesian procedures are characterized and illustrated as well. This effort extends earlier work of Perlman and Rasmussen and of Neff and Strawderman.

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